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Complete integrability of two-coupled discrete modified Korteweg–de Vries equations

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Abstract

We report that two-coupled discrete modified Korteweg–de Vries equations (2-dmKdV) governed by nonlinear partial differential–difference equations are completely integrable. We derive Lax matrices for 2-dmKdV and also show that it admits a sequence of generalized (non-point) symmetries and polynomial conserved densities establishing its complete integrability. Also exact solutions of it expressible in terms of Hyperbolic and Jacobian elliptic functions have been derived.

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1. Introduction

The study of discrete nonlinear systems governed by both ordinary and partial differential–difference (including lattice equations) and pure difference equations has drawn much attention in recent years particularly from the point of complete integrability [1–15]. Several analytical techniques have been devised to determine whether or not the given nonlinear differential–difference equation is completely integrable. Among them the continuous transformation group theory and Lax pair technique developed, respectively, by Sophus Lie and Lax play a significant role [16–19]. Given a nonlinear evolution equation possessing mathematical structures, both algebraic and analytic related with its integrability, how to find its discrete analogue preserving integrability structures is one of the current topics of researchers in nonlinear equations. As a result of the concerted efforts of the several research groups, a discrete analogue of different nonlinear partial differential equations including Korteweg–de Vries, modified Korteweg–de Vries, nonlinear Schrödinger equation, etc has been derived preserving the integrability properties of their continuous counter parts [2, 3, 20–24]. Recently, a similar effort to derive the discrete analogue of coupled nonlinear partial differential equations with

(1+1) dimensions possessing solitons has been extended [25–27]. For example, Ablowitz *et al* [25] have considered a system of N-coupled nonlinear Schrödinger equations

$$i\frac{\partial}{\partial t}\chi_k = c_k\frac{\partial^2}{\partial x^2}\chi_k + 2\alpha_{kk}|\chi_k|^2\chi_k + 2\sum_{l=1}^{N}\alpha_{kl}|\chi_l|^2\chi_k,$$
$$l \neq k, \quad \alpha_{kl} = \alpha_{lk}, \quad k, l = 1, 2, \ldots, N \tag{1}$$

and derived a discrete analogue

$$i\frac{\mathrm{d}}{\mathrm{d}t}q_n^{(j)} = \left(q_{n+1}^{(j)} + q_{n-1}^{(j)} - 2q_n^{(j)}\right) + \sum_{k=1}^{N}\left|q_n^{(j)}\right|^2\left(q_{n+1}^{(j)} + q_{n-1}^{(j)}\right), \qquad j = 1, 2, \ldots, N \tag{2}$$

which preserves integrability properties of its continuous counter part. In this paper, we show that the 2-coupled discrete modified Korteweg–de Vries equation (2-dmKdV), namely

$$\frac{\partial a_n}{\partial t} = \left(1 + a_n^2 + b_n^2\right)(a_{n+1} - a_{n-1}), \qquad a_{n-1} = a(n-1, t), \qquad a_n = a(n, t), \quad \text{etc}, \tag{3a}$$

$$\frac{\partial b_n}{\partial t} = \left(1 + a_n^2 + b_n^2\right)(b_{n+1} - b_{n-1}) \tag{3b}$$

which can be viewed as a discrete version of the 2-coupled modified Korteweg–de Vries equation [28],

$$u_{1\tau} + 6\left(u_1^2 + u_2^2\right)u_{1x} + u_{1xxx} = 0, \tag{4a}$$

$$u_{2\tau} + 6\left(u_1^2 + u_2^2\right)u_{2x} + u_{2xxx} = 0, \tag{4b}$$

is completely integrable. It is appropriate to mention here that Tsuchida and Wadati [29] have shown that the above 2-coupled modified Korteweg–de Vries equations is solvable by the inverse scattering transform technique and has multi-soliton solutions while Sahadevan and Kannagi [30] have shown that it admits infinitely many generalized symmetries, polynomial conservation laws and a recursion operator establishing its integrability.

We wish to mention that equation (4) can be achieved from equation (3) through the following limiting procedure:

$$\begin{aligned}
a_n &= \delta u_1\left((n+2t)\delta, \tfrac{1}{3}\delta^3 t\right) + O(\delta^2),\\
&\equiv \delta u_1(x, \tau) + O(\delta^2),\\
b_n &= \delta u_2\left((n+2t)\delta, \tfrac{1}{3}\delta^3 t\right) + O(\delta^2),\\
&\equiv \delta u_2(x, \tau) + O(\delta^2),\\
a_{n\pm1} &= \delta u_1(x \pm \delta, \tau) + O(\delta^2),\\
b_{n\pm1} &= \delta u_2(x \pm \delta, \tau) + O(\delta^2).
\end{aligned}$$

Substituting the above into equation (3), we obtain the 2-coupled modified Korteweg–de Vries equation,

$$\left(u_{1\tau} + 6\left(u_1^2 + u_2^2\right)u_{1x} + u_{1xxx}\right)\delta^4 + O(\delta^5) = 0, \tag{5a}$$

$$\left(u_{2\tau} + 6\left(u_1^2 + u_2^2\right)u_{2x} + u_{2xxx}\right)\delta^4 + O(\delta^5) = 0. \tag{5b}$$

This paper is organized as follows. In section 2, we show that 2-dmKdV admits a Lax representation, indicating that it is integrable in the sense of Lax. In section 3, we show explicitly that the 2-dmKdV possesses a sequence of generalized symmetries and polynomial conserved densities. In section 4, we derive some exact solutions of 2-dmKdV expressible in terms of Hyperbolic and Jacobian elliptic functions. In section 5, we give a summary of our results.

2. Lax representation

An autonomous nonlinear partial differential–difference equation (PDΔE) with two independent variables (one discrete and the other continuous) is an equation of form

$$\frac{\partial \mathbf{u}_n}{\partial t} = \mathbf{F}(\ldots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots), \tag{6}$$

where $\mathbf{u}_n$ and $\mathbf{F}$ are the vector-valued functions. The Lax representation of a scalar PDΔE, that is for

$$\frac{\partial}{\partial t} a(n,t) = F(a(n-1,t), a(n,t), a(n+1,t)) \qquad \text{or} \qquad \frac{\partial a_n}{\partial t} = F(a_{n-1}, a_n, a_{n+1}),$$

can be constructed in the following manner. Consider a linear system

$$\mathbf{\Phi}_{n+1}(t,\lambda) = L_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}_n(t,\lambda) = M_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda) \tag{7}$$

or equivalently,

$$\begin{bmatrix} \phi_{1_{n+1}}(t,\lambda) \\ \phi_{2_{n+1}}(t,\lambda) \end{bmatrix} = \begin{bmatrix} L_{11n}(t,\lambda) & L_{12n}(t,\lambda) \\ L_{21n}(t,\lambda) & L_{22n}(t,\lambda) \end{bmatrix} \begin{bmatrix} \phi_{1_n}(t,\lambda) \\ \phi_{2_n}(t,\lambda) \end{bmatrix},$$

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\,\phi_{1_n}(t,\lambda) \\ \frac{\mathrm{d}}{\mathrm{d}t}\,\phi_{2_n}(t,\lambda) \end{bmatrix} = \begin{bmatrix} M_{11n}(t,\lambda) & M_{12n}(t,\lambda) \\ M_{21n}(t,\lambda) & M_{22n}(t,\lambda) \end{bmatrix} \begin{bmatrix} \phi_{1_n}(t,\lambda) \\ \phi_{2_n}(t,\lambda) \end{bmatrix},$$

where λ is the spectral parameter and $L_{ijn}(t,\lambda)$ and $M_{ijn}(t,\lambda)$ are the functions of a_n and its shifts. The compatibility condition of the linear system (7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} L_n + L_n M_n - M_{n+1} L_n = 0. \tag{8}$$

It is straightforward to derive the Lax matrices for the scalar dmKdV

$$\frac{\partial a_n}{\partial t} = \left(1 + a_n^2\right)(a_{n+1} - a_{n-1})$$

satisfying equation (8). The explicit form of Lax matrices is

$$L_n = \begin{bmatrix} \lambda & a_n \\ -a_n & 1/\lambda \end{bmatrix}, \qquad M_n = \begin{bmatrix} \lambda^2 + a_n a_{n-1} & a_n\lambda + \frac{a_{n-1}}{\lambda} \\ -a_{n-1}\lambda - \frac{a_n}{\lambda} & \frac{1}{\lambda^2} + a_n a_{n-1} \end{bmatrix}.$$

For a 2-coupled PDΔE with two independent variables (one discrete and the other continuous), that is for

$$\frac{\partial a_n}{\partial t} = F_1(a_{n-1}, b_{n-1}, a_n, b_n, a_{n+1}, b_{n+1}),$$
$$\frac{\partial b_n}{\partial t} = F_2(a_{n-1}, b_{n-1}, a_n, b_n, a_{n+1}, b_{n+1}),$$

the associated linear equations read

$$\mathbf{\Phi}_{n+1}(t,\lambda) = L_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{\Phi}_n(t,\lambda) = M_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda),$$

where $\mathbf{\Phi}_n(t,\lambda) = (\phi_{1_n}(t,\lambda), \phi_{2_n}(t,\lambda), \phi_{3_n}(t,\lambda), \phi_{4_n}(t,\lambda))^T$, $L_n(t,\lambda)$ and $M_n(t,\lambda)$ are 4×4 matrices with entries, $L_{ijn}(t,\lambda)$ and $M_{ijn}(t,\lambda)$ are the functions of a_n, b_n and their shifts. The explicit form of the Lax matrices $L_n(t,\lambda)$ and $M_n(t,\lambda)$ can be derived by extending a well-known procedure devised by Ablowitz, Kaup, Newell and Segur (AKNS) for nonlinear partial differential equations [19]. More precisely, for a given suitable matrix $L_n(t,\lambda)$, the matrix $M_n(t,\lambda)$ can be derived by expanding its entries as a polynomial in the spectral parameter λ or $\frac{1}{\lambda}$ satisfying the Lax equation (8).

To derive the Lax matrices for 2-dmKdV we first fix the entries of matrix $L_n(t,\lambda)$ as

$$L_n(t,\lambda) = \begin{bmatrix} \lambda & 0 & a_n & b_n \\ 0 & \lambda & b_n & -a_n \\ -a_n & -b_n & \frac{1}{\lambda} & 0 \\ -b_n & a_n & 0 & \frac{1}{\lambda} \end{bmatrix}$$

and then expand each entry of the matrix $M_n(t,\lambda)$ as a polynomial in $\left(\frac{1}{\lambda}\right)^l$, $l = -2, -1, 0, 1, 2$. Proceeding further along with the matrix $L_n(t,\lambda)$, we find that the compatibility condition given in equation (8) satisfied for the matrices

$$M_n(t,\lambda) = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ -M_{12} & M_{11} & M_{14} & -M_{13} \\ M_{31} & -M_{32} & M_{33} & M_{34} \\ -M_{32} & M_{31} & -M_{34} & M_{33} \end{bmatrix},$$

where

$$M_{11} = \lambda^2 + a_n a_{n-1} + b_n b_{n-1}, \qquad M_{12} = a_n b_{n-1} - b_n a_{n-1},$$
$$M_{13} = a_n\lambda + \frac{a_{n-1}}{\lambda}, \qquad M_{14} = b_n\lambda + \frac{b_{n-1}}{\lambda},$$
$$M_{31} = a_{n-1}\lambda + \frac{a_n}{\lambda}, \qquad M_{32} = b_{n-1}\lambda + \frac{b_n}{\lambda}$$
$$M_{33} = \frac{1}{\lambda^2} + a_n a_{n-1} + b_n b_{n-1}, \qquad M_{34} = a_n b_{n-1} - b_n a_{n-1}.$$

Thus, we infer that the 2-dmKdV system (3) is integrable in the sense of Lax.

3. Generalized symmetries and polynomial conserved densities

3.1. Generalized symmetries

It is easy to check that 2-dmKdV given in equation (3) is not invariant under scaling symmetry. By introducing an auxiliary parameter K we find that the 2-dmKdV given by

$$\frac{\partial a_n}{\partial t} = \left(K + a_n^2 + b_n^2\right)(a_{n+1} - a_{n-1}), \tag{9a}$$

$$\frac{\partial b_n}{\partial t} = \left(K + a_n^2 + b_n^2\right)(b_{n+1} - b_{n-1}) \tag{9b}$$

is invariant under scaling (dilation) symmetry

$$(t, a_n, b_n, K) \to \left(\lambda^{-1}t, \lambda^{\frac{1}{2}}a_n, \lambda^{\frac{1}{2}}b_n, \lambda K\right), \tag{10}$$

where λ is an arbitrary parameter. Let us assume that equation (9) is invariant under a one-parameter (ϵ) continuous non-point transformations

$$n^* = n, \qquad t^* = t, \qquad a_n^* = a_n + \epsilon G_i^{(1)}(n) + O(\epsilon^2),$$
$$b_n^* = b_n + \epsilon G_i^{(2)}(n) + O(\epsilon^2), \qquad i = 1, 2, \ldots, \tag{11}$$

where

$$G_i^{(1)}(n) = G_i^{(1)}(\ldots, b_{n-1}, a_{n-1}, a_n, b_n, a_{n+1}, b_{n+1}, \ldots),$$
$$G_i^{(2)}(n) = G_i^{(2)}(\ldots, b_{n-1}, a_{n-1}, a_n, b_n, a_{n+1}, b_{n+1}, \ldots)$$

provided that a_n and b_n satisfy equation (9). For clarity, we denote $\mathbf{G}_i(n) = (G_i^{(1)}(n), G_i^{(2)}(n))^T$ and the subscript i represents the ith order non-point or generalized symmetry. Consequently, we obtain the following invariant equations:

$$\frac{\partial G_i^{(1)}(n)}{\partial t} = \left(2a_n G_i^{(1)}(n) + 2b_n G_i^{(2)}(n)\right)[a_{n+1} - a_{n-1}] + \left(G_i^{(1)}(n+1) - G_i^{(1)}(n-1)\right)\left[K + a_n^2 + b_n^2\right], \tag{12a}$$

$$\frac{\partial G_i^{(2)}(n)}{\partial t} = \left(2a_n G_i^{(1)}(n) + 2b_n G_i^{(2)}(n)\right)[b_{n+1} - b_{n-1}] + \left(G_i^{(2)}(n+1) - G_i^{(2)}(n-1)\right)\left[K + a_n^2 + b_n^2\right]. \tag{12b}$$

The invariant equations (12a) and (12b) can be solved for the generalized symmetry $\mathbf{G}_i(n) = \left(G_i^{(1)}(n), G_i^{(2)}(n)\right)^T$ in more than one way [18, 31–34, 37]. We show below how to derive the generalized symmetries of equation (9) through an algorithmic procedure developed by Hereman and his collaborators [32]. Basically, the Hereman's algorithmic procedure is based on the concept of weights and ranks. To start with, we briefly explain the concept of weights and ranks. The weight, w, of a variable is defined as the exponent in the scaling parameter λ which multiplies the variable. Weights of the dependent variables are non-negative, rational and independent of n. Similarly, the rank of a monomial is defined as the total weight of the monomial. An expression is said to be uniform in rank if all its terms have the same rank. We wish to mention that Hereman and his collaborators have developed a *Mathematica* software (known as **InvariantsSymmetries.m**) to derive (i) generalized symmetries, (ii) conserved densities for partial differential equations and differential–difference equations. In this paper, we have computed the generalized symmetries and conserved densities manually.

We set $w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = 1$. From equation (9) we see

$$w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) + w(a_n) = w(K) + w(a_n) = 3w(a_n) = w(a_n) + 2w(b_n),$$

$$w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) + w(b_n) = w(K) + w(b_n) = 3w(b_n) = 2w(a_n) + w(b_n)$$

and so

$$w(a_n) = \tfrac{1}{2}, \qquad w(b_n) = \tfrac{1}{2}, \qquad w(K) = 1$$

and hence (9a) and (9b) are of the same rank $\frac{3}{2}$. Hereafter, we use the more compact notation

$$a_n = a, \quad b_n = b, \quad a_{n-1} = \underline{a}, \quad b_{n-1} = \underline{b}, \quad a_{n-2} = \underline{\underline{a}}, \quad b_{n-2} = \underline{\underline{b}}, \quad a_{n-3} = \underline{\underline{\underline{a}}},$$

$$b_{n-3} = \underline{\underline{\underline{b}}}, \quad a_{n+1} = \overline{a}, \quad b_{n+1} = \overline{b}, \quad a_{n+2} = \overline{\overline{a}}, \quad b_{n+2} = \overline{\overline{b}}, \quad a_{n+3} = \overline{\overline{\overline{a}}}, \quad b_{n+3} = \overline{\overline{\overline{b}}}, \quad \ldots .$$

From equation (12) we see that

$$\mathbf{G}_2(n) = \begin{pmatrix} G_2^{(1)}(n) \\ G_2^{(2)}(n) \end{pmatrix} = \begin{pmatrix} (K + a^2 + b^2)(\overline{a} - \underline{a}) \\ (K + a^2 + b^2)(\overline{b} - \underline{b}) \end{pmatrix} \tag{13}$$

is a trivial generalized symmetry with rank $\left(\frac{3}{2}, \frac{3}{2}\right)$, and therefore the next non-trivial generalized symmetry $\mathbf{G}_3(n) = \binom{G_3^{(1)}(n)}{G_3^{(2)}(n)}$ must have rank $\left(\frac{5}{2}, \frac{5}{2}\right)$. With this in mind, we first form a monomial in a_n and b_n of rank $\left(\frac{5}{2}, \frac{5}{2}\right)$ that leads to a set:

$$\mathcal{L} = \{a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3, a^4, a^3b, a^2b^2, ab^3, b^4, a^5, a^4b, a^3b^2, a^2b^3, ab^4, b^5, Ka, Kb, Ka^2, Kab, Kb^2, Ka^3, Ka^2b, Kab^2, Kb^3, K^2a, K^2b\}.$$

Then the necessary partial derivatives with respect to t in each monomial of $\mathcal{L}$ along with equation (9) give the following:

$$\frac{\mathrm{d}^0}{\mathrm{d}t^0}(a^5) = a^5, \qquad \frac{\mathrm{d}^0}{\mathrm{d}t^0}(a^4b) = a^4b, \qquad \frac{\mathrm{d}^0}{\mathrm{d}t^0}(a^3b^2) = a^3b^2,$$
$$\frac{\mathrm{d}^0}{\mathrm{d}t^0}(a^2b^3) = a^2b^3, \qquad \frac{\mathrm{d}^0}{\mathrm{d}t^0}(ab^4) = ab^4, \qquad \frac{\mathrm{d}^0}{\mathrm{d}t^0}(b^5) = b^5,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 3a^2\dot{a} = 3a^2(K + a^2 + b^2)(\overline{a} - \underline{a})$$
$$= 3Ka^2\overline{a} - 3Ka^2\underline{a} + 3a^4\overline{a} - 3a^4\underline{a} + 3a^2b^2\overline{a} - 3a^2b^2\underline{a}, \text{ etc},$$

and a set

$$\begin{aligned}\mathcal{M} = \{&K^2a, K^2\overline{a}, K^2\underline{a}, K^2\overline{\overline{a}}, K^2\underline{\underline{a}}, K^2b, K^2\overline{b}, K^2\underline{b}, K^2\overline{\overline{b}}, K^2\underline{\underline{b}}, Ka^3, Kb^3, Ka^2b,\\
&Kab^2, Ka^2\overline{a}, Ka^2\underline{a}, Ka^2\overline{\overline{a}}, Ka^2\underline{\underline{a}}, Ka^2\overline{b}, Ka^2\underline{b}, Ka^2\overline{\overline{b}}, Ka^2\underline{\underline{b}}, Ka\overline{a}^2,\\
&Ka\underline{a}^2, Ka\overline{b}^2, Ka\underline{b}^2, K\overline{a}b^2, K\underline{a}b^2, K\overline{\overline{a}}b^2, K\underline{\underline{a}}b^2, Kb^2\overline{b}, Kb^2\underline{b}, Kb^2\overline{\overline{b}}, Kb^2\underline{\underline{b}},\\
&K\overline{a}^2b, K\underline{a}^2b, Kb\overline{b}^2, Kb\underline{b}^2, K\overline{a}^2\overline{\overline{a}}, K\overline{\overline{a}}\,\overline{b}^2, K\overline{a}^2\overline{\overline{b}}, K\overline{b}^2\overline{\overline{b}}, K\overline{a}^2\underline{b}, K\underline{\underline{a}}\,\underline{b}^2, K\underline{a}^2\underline{\underline{a}},\\
&K\underline{b}^2\underline{\underline{b}}, K\underline{a}^2\underline{\underline{b}}_{-2}, Ka\overline{a}b, Ka\underline{a}b, Kab\overline{b}, Kab\underline{b}, Ka\overline{a}\underline{a}, Kb\overline{b}\underline{b}, Ka\overline{a}\overline{b}, Ka\overline{a}\underline{b},\\
&Ka\underline{a}\overline{b}, Ka\underline{a}\,\underline{b}, K\overline{a}b\overline{b}, K\overline{a}b\underline{b}, K\underline{a}b\overline{b}, K\underline{a}b\underline{b}, a^5, b^5, a^4b, a^3b^2, a^2b^3, ab^4, a^4\overline{a},\\
&a^4\underline{a}, a^4\overline{b}, a^4\underline{b}, a^3\overline{a}b, a^3\underline{a}b, a^3b\overline{b}, a^3b\underline{b}, \overline{a}b^4, \underline{a}b^4, b^4\overline{b}, b^4\underline{b}, a^2\overline{a}b^2, a^2\underline{a}b^2, a^2b^2\overline{b},\\
&a^2b^2\underline{b}, a\overline{a}b^3, a\underline{a}b^3, ab^3\overline{b}, ab^3\underline{b}, a^3\overline{a}^2, a^3\underline{a}^2, a^3\overline{b}^2, a^3\underline{b}^2, a^3\overline{a}\underline{a}, a^3\overline{a}\overline{b}, a^3\overline{a}\underline{b},\\
&a^3\underline{a}\,\overline{b}, a^3\underline{a}\,\underline{b}, \overline{a}^2b^3, b^3\overline{b}^2, \underline{a}^2b^3, b^3\underline{b}^2, b^3\overline{b}\underline{b}, \overline{a}b^3\overline{b}, \overline{a}b^3\underline{b}, \underline{a}b^3\overline{b}, \underline{a}b^3\underline{b}, a^2\overline{a}^2b, a^2b\overline{b}^2,\\
&a^2\underline{a}^2b, a^2b\underline{b}^2, a^2b\overline{b}\underline{b}, a^2\overline{a}b\overline{b}, a^2\overline{a}b\underline{b}, a^2\underline{a}b\overline{b}, a^2\underline{a}b\underline{b}, a\overline{a}^2b^2, a\underline{a}^2b^2, ab^2\overline{b}^2,\\
&ab^2\underline{b}^2, a\overline{a}\underline{a}b^2, a\overline{a}b^2\overline{b}, a\overline{a}b^2\underline{b}, a\underline{a}b^2\overline{b}, a\underline{a}b^2\underline{b}, b^2\underline{b}^2\underline{\underline{b}}, \underline{a}^2b^2\underline{\underline{b}}, a^2\overline{a}^2\overline{\overline{b}}, \underline{\underline{a}}b^2\underline{b}^2, a^2\underline{a}^2\underline{\underline{a}},\\
&b^2\overline{b}^2\overline{\overline{b}}, \overline{\overline{a}}b^2\overline{b}^2, \underline{a}^2\underline{\underline{a}}b^2, \overline{a}^2\overline{\overline{a}}b^2, a^2\underline{a}^2\underline{\underline{b}}, a^2\overline{b}^2\overline{\overline{b}}, a^2\underline{\underline{a}}\,\underline{b}^2, a^2\underline{b}^2\underline{\underline{b}}, a^2\overline{a}^2\overline{\overline{a}}, a^2\overline{\overline{a}}\,\overline{b}^2, \overline{a}^2b^2\overline{\overline{b}}\}.\end{aligned}$$

Thus, the most general form of the non-trivial generalized symmetry $\mathbf{G}_3(n) = \left(G_3^{(1)}(n), G_3^{(2)}(n)\right)^T$ is a linear combination of the elements in the set $\mathcal{M}$. Substituting the linear combination for $G_3^{(1)}(n)$ and $G_3^{(2)}(n)$ in equations (12*a*) and (12*b*) along with 2-dmKdV, we find that it satisfies identically for the following forms:

$$\begin{aligned}G_3^{(1)} = {}&(K + a^2 + b^2)[(K + \overline{a}^2 + \overline{b}^2)\overline{\overline{a}} - (K + \underline{a}^2 + \underline{b}^2)\underline{\underline{a}}\\
&+ (\overline{a}^2 - \underline{a}^2 - \overline{b}^2 + \underline{b}^2)a + 2b(\overline{a} - \underline{a})(\overline{b} + \underline{b})],\end{aligned} \tag{14a}$$

$$\begin{aligned}G_3^{(2)} = {}&(K + a^2 + b^2)[(K + \overline{a}^2 + \overline{b}^2)\overline{\overline{b}} - (K + \underline{a}^2 + \underline{b}^2)\underline{\underline{b}}\\
&+ (\overline{b}^2 - \underline{b}^2 - \overline{a}^2 + \underline{a}^2)a + 2a(\overline{a} + \underline{a})(\overline{b} - \underline{b})].\end{aligned} \tag{14b}$$

Note that when $K = 1$, we obtain the generalized symmetries of 2-dmKdV, equation (3). Proceeding as above, we obtain the next order non-trivial generalized symmetry $\mathbf{G}_4(n) = \left(G_4^{(1)}(n), G_4^{(2)}(n)\right)^T$ with rank $\left(\frac{7}{2}, \frac{7}{2}\right)$. The explicit forms of $G_4^{(1)}(n)$ and $G_4^{(2)}(n)$ with $K = 1$ are:

$$\begin{aligned}G_4^{(1)} = {}&(1 + a^2 + b^2)[(1 + \overline{a}^2 + \overline{b}^2)((1 + \overline{\overline{a}}^2 + \overline{\overline{b}}^2)\overline{\overline{\overline{a}}} + \overline{a}\,\overline{\overline{a}}^2 + 2\overline{\overline{a}}(a\overline{a} + b\overline{b} + b\underline{b} + \overline{b}\,\overline{\overline{b}})\\
&- \overline{a}\overline{\overline{b}}^2 - 2\overline{\overline{b}}(a\overline{b} + \underline{a}\,b - \overline{a}\,b)) - (1 + \underline{a}^2 + \underline{b}^2)((1 + \underline{\underline{a}}^2 + \underline{\underline{b}}^2)\underline{\underline{\underline{a}}} + \underline{a}\,\underline{\underline{a}}^2 - \underline{a}\,\underline{\underline{b}}^2\end{aligned}$$

$$
\begin{aligned}
&+ 2\underline{\underline{a}}(a\underline{a} + b\overline{b} + b\underline{b} + \underline{b}\,\underline{\underline{b}}) - 2\underline{\underline{b}}(a\underline{b} + \overline{a}\,b - \underline{a}\,b)) + (a^2 - b^2)(\overline{a}^3 - 3\overline{a}\overline{b}^2 \\
&+ 3\underline{a}\,\underline{b}^2 - \underline{a}^3) + (1 + a^2 + 3\,b^2)(\overline{a}^2\underline{a} + 2\overline{a}\overline{b}\underline{b} - \overline{a}\underline{a}^2 + \overline{a}\underline{b}^2 - \underline{a}\overline{b}^2 - 2\underline{a}\overline{b}\underline{b}) \\
&+ 2a\,b(\overline{a}^2\underline{b} - \underline{a}^2\overline{b} + \underline{b}^3 - \overline{b}^3 + \overline{b}\underline{b}^2 - \overline{b}^2\underline{b} + 3\overline{a}^2\overline{b} - 3\underline{a}^2\underline{b} + 2\overline{a}\underline{a}\,\underline{b} - 2\overline{a}\underline{a}\overline{b})],
\end{aligned}
$$

$$
\begin{aligned}
G_4^{(2)} = {} & (1 + a^2 + b^2)[(1 + \overline{a}^2 + \overline{b}^2)((1 + \overline{\overline{a}}^2 + \overline{\overline{b}}^2)\overline{\overline{b}} + \overline{b}\,\overline{\overline{b}}^2 + 2\overline{\overline{b}}(a\overline{a} + a\underline{a} + \overline{a}\,\overline{\overline{a}} + b\overline{b}) \\
&- \overline{\overline{a}}^2\overline{b} - 2\overline{\overline{a}}(a\underline{b} + \overline{a}\,b - a\overline{b})) - (1 + \underline{a}^2 + \underline{b}^2)((1 + \underline{\underline{a}}^2 + \underline{\underline{b}}^2)\underline{\underline{b}} + \underline{b}\,\underline{\underline{b}}^2 - \underline{\underline{a}}^2\underline{b} \\
&+ 2\underline{\underline{b}}(a\overline{a} + a\underline{a} + \underline{a}\,\underline{\underline{a}} + b\underline{b}) - 2\underline{\underline{a}}(\underline{a}\,b + a\overline{b} - a\underline{b})) + (a^2 - b^2)(\overline{b}^3 - 3\overline{a}^2\overline{b} \\
&+ 3\underline{a}^2\underline{b} - \underline{b}^3) + (1 + 3\,a^2 + b^2)(\overline{b}^2\underline{b} + 2\overline{a}\underline{a}\overline{b} - \overline{b}\underline{b}^2 + \underline{a}^2\overline{b} - \overline{a}^2\underline{b} - 2\overline{a}\underline{a}\,\underline{b}) \\
&+ 2a\,b(\underline{a}\overline{b}^2 - \overline{a}\underline{b}^2 + \underline{a}^3 - \overline{a}^3 + \overline{a}\underline{a}^2 - \overline{a}^2\underline{a} + 3\overline{a}\overline{b}^2 - 3\underline{a}\,\underline{b}^2 + 2\underline{a}\overline{b}\underline{b} - 2\overline{a}\overline{b}\underline{b})].
\end{aligned}
$$

In a similar manner, one can derive a sequence of next higher order generalized symmetries $\{\mathbf{G}_i(n)\} = \left\{\left(G_i^{(1)}(n), G_i^{(2)}(n)\right)^T\right\}$ with rank $\left(\frac{2i-1}{2}, \frac{2i-1}{2}\right)$, $i = 5, 6, \ldots$. We have checked that the obtained sequences of generalized symmetries $\{\mathbf{G}_i(n)\}$ also satisfy the following relation:

$$[\mathbf{G}_i(n), \mathbf{G}_j(n)] = \mathbf{G}_i(n)'[\mathbf{G}_j(n)] - \mathbf{G}_j(n)'[\mathbf{G}_i(n)] = 0 \quad \forall i, \ j,$$

where $\mathbf{G}_i(n)'[\mathbf{G}_j(n)]$ is the Fréchet derivative of $\mathbf{G}_i(n)$ in the direction of $\mathbf{G}_j(n)$. Thus, the obtained sequences of generalized symmetries commute each other.

3.2. Polynomial conserved densities and fluxes

A scalar function $\rho_n(\mathbf{u}_n)$ is a conserved density of (6) if there exists a scalar function $J_n(\mathbf{u}_n)$ called the associated flux, such that

$$\frac{\partial \rho_n}{\partial t} + \Delta J_n = 0 \tag{15}$$

is satisfied on the solutions of (6). Here $\Delta J_n = (E - I)J_n = J_{n+1} - J_n$. To derive conserved densities with different ranks, we use the algorithmic procedure (homotopy operator) of Hereman and his collaborators [35, 36]. For rank 2 as usual we form monomials of a and b which give the list $\mathcal{L}_1 = \{a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3, a^4, a^3b, a^2b^2, ab^3, b^4, Ka, Kb, Ka^2, Kab, Kb^2\}$. Introducing the necessary t derivatives in each monomial of $\mathcal{L}_1$ leads to a set

$$
\begin{aligned}
\mathcal{M}_1 = {} & \{a^4, a^3b, a^2b^2, ab^3, b^4, Ka^2, Kab, Kb^2, Ka\overline{a}, Kb\overline{b}, Ka\overline{b}, Ka\underline{b}, a^3\overline{a}, a^3\underline{a}, a^3\overline{b}, \\
& a^3\underline{b}, \overline{a}b^3, \underline{a}b^3, b^3\overline{b}, b^3\underline{b}, a^2\overline{a}b, a^2\underline{a}b, a^2b\overline{b}, a^2b\underline{b}, a\overline{a}b^2, a\underline{a}b^2, ab^2\overline{b}, ab^2\underline{b}\}.
\end{aligned}
$$

Thus the most general form of the conserved density of rank 2 will be

$$
\begin{aligned}
\rho_n^{(2)} = {} & c_1a^4 + c_2a^3b + c_3a^2b^2 + c_4ab^3 + c_5b^4 + c_6Ka^2 + c_7Kab + c_8Kb^2 \\
& + c_9Ka\overline{a} + c_{10}Kb\overline{b} + c_{11}Ka\overline{b} + c_{12}Ka\underline{b} + c_{13}a^3\overline{a} + c_{14}a^3\underline{a} + c_{15}a^3\overline{b} \\
& + c_{16}a^3\underline{b} + c_{17}\overline{a}b^3 + c_{18}\underline{a}b^3 + c_{19}b^3\overline{b} + c_{20}b^3\underline{b} + c_{21}a^2\overline{a}b + c_{22}a^2\underline{a}b \\
& + c_{23}a^2b\overline{b} + c_{24}a^2b\underline{b} + c_{25}a\overline{a}b^2 + c_{26}a\underline{a}b^2 + c_{27}ab^2\overline{b} + c_{28}ab^2\underline{b},
\end{aligned}
$$

where c_i, $i = 1, 2, \ldots, 28$ are constants to be determined. Using the above along with (3) in (15) we obtain the conserved density $\rho_n^{(2)}$ with rank 2

$$\rho_n^{(2)} = Ka\overline{a} + Kb\overline{b} \tag{16}$$

and the associated fluxes $J_n^{(2)}$ as

$$J_n^{(2)} = K^3 - K(K + a^2 + b^2)(K + \underline{a}\overline{a} + \underline{b}\overline{b}). \tag{17}$$

Proceeding as above, here again, we obtain the conserved density $\rho_n^{(3)}$ with rank 3

$$\begin{aligned}\rho_n^{(3)} &= \tfrac{1}{2}Ka^2\overline{a}^2 + K^2a\overline{\overline{a}} + Ka\overline{a}^2\overline{\overline{a}} + \tfrac{1}{2}Kb^2\overline{b}^2 + K^2b\overline{\overline{b}} + Kb\overline{b}^2\overline{\overline{b}} \\ &\quad - \tfrac{1}{2}K\overline{a}^2b^2 - \tfrac{1}{2}Ka^2\overline{b}^2 + Ka\overline{\overline{a}}\,\overline{b}^2 + K\overline{a}^2b\overline{\overline{b}} + 2Ka\overline{a}b\overline{b}\end{aligned} \tag{18}$$

and the associated fluxes $J_n^{(3)}$ as

$$\begin{aligned}J_n^{(3)} &= a(K + a^2 + b^2)(K\underline{a}\overline{b}^2 - K^2\overline{a} - K\overline{a}^2\underline{a} - 2K\overline{a}\overline{b}\underline{b}) \\ &\quad + b(K + a^2 + b^2)(K\overline{a}^2\underline{b} - K^2\overline{b} - K\overline{b}^2\underline{b} - 2K\underline{a}\overline{a}\overline{b}) \\ &\quad - (K + a^2 + b^2)(K^2\underline{a}\overline{\overline{a}} + K^2\underline{b}\overline{\overline{b}} + K\underline{a}\overline{a}^2\overline{\overline{a}} + K\underline{b}\overline{b}^2\overline{\overline{b}} + K\underline{a}\,\overline{\overline{a}}\,\overline{b}^2 + K\overline{a}^2\underline{b}\overline{\overline{b}}).\end{aligned} \tag{19}$$

In a similar manner, one can generate a sequence of higher order conserved densities $\rho_n^{(k)}, k = 4, 5, \ldots$ with rank $4, 5, \ldots$ along with the fluxes which involves lengthy expressions and so the details are omitted here.

4. Some exact solutions of 2-dmKdV

It is known that the scalar dmKdV

$$\frac{\partial a_n}{\partial t} = \left(1 + a_n^2\right)(a_{n+1} - a_{n-1})$$

admits an exact solution having the form

$$a(n, t) = i \tanh(k) \tanh(kn + 2\tanh(k)t),$$

where k is an arbitrary constant. In this section, we show that 2-dmKdV also admits an exact solution expressible in terms of hyperbolic and Jacobian elliptic functions. Let us assume that 2-dmKdV admits an exact solution of form

$$a(n, t) = A_1 + A_2 \tanh(kn + \omega t), \qquad b(n, t) = B_1 + B_2 \tanh(kn + \omega t) \tag{20}$$

with k being the wave number, ω being the angular frequency and A_1, A_2, B_1 and B_2 being the constants to be determined. Substituting (20) into 2-dmKdV and then equating the coefficients of $\tanh^l(kn + \omega t)$, $l = 0, 1, 2$ to zero, we obtain the following equations:

$$\begin{aligned}&A_1A_2 + B_1B_2 = 0, \\ &\omega \tanh(k) + 2A_2^2 + 2B_2^2 = 0, \\ &-\omega + 2\tanh(k) + 2\tanh(k)A_1^2 + 2\tanh(k)B_1^2 = 0.\end{aligned}$$

Solving the above equations consistently we find

$$A_1 = -\mathrm{i}B_2\sqrt{\frac{\left(A_2^2 + B_2^2 + \tanh^2(k)\right)}{\left(A_2^2 + B_2^2\right)\tanh^2(k)}}, \qquad B_1 = \mathrm{i}A_2\sqrt{\frac{\left(A_2^2 + B_2^2 + \tanh^2(k)\right)}{\left(A_2^2 + B_2^2\right)\tanh^2(k)}},$$

$$\omega = -\frac{2\left(A_2^2 + B_2^2\right)}{\tanh(k)}$$

and so

$$a(n, t) = -\mathrm{i}B_2\sqrt{\frac{\left(A_2^2 + B_2^2 + \tanh^2(k)\right)}{\left(A_2^2 + B_2^2\right)\tanh^2(k)}} + A_2\tanh(kn + \omega t), \tag{21a}$$

$$b(n, t) = \mathrm{i}A_2\sqrt{\frac{\left(A_2^2 + B_2^2 + \tanh^2(k)\right)}{\left(A_2^2 + B_2^2\right)\tanh^2(k)}} + B_2\tanh(kn + \omega t). \tag{21b}$$

Proceeding in a similar manner, we find that 2-dmKdV admits an exact solution expressible in terms of Jacobian elliptic functions. They are

$$a_n = -\mathrm{i}B_2\sqrt{\frac{A_2^2 + B_2^2 + m^2(\mathrm{sn}(k,m))^2}{\left(A_2^2 + B_2^2\right)m^2(\mathrm{sn}(k,m))^2}} + A_2\mathrm{sn}(kn + \omega t, m), \tag{22a}$$

$$b_n = \mathrm{i}A_2\sqrt{\frac{A_2^2 + B_2^2 + m^2(\mathrm{sn}(k,m))^2}{\left(A_2^2 + B_2^2\right)m^2(\mathrm{sn}(k,m))^2}} + B_2\mathrm{sn}(kn + \omega t, m), \tag{22b}$$

where $\omega = -\frac{2(A_2^2+B_2^2)}{m^2\mathrm{sn}(k,m)}$ and $m \neq 0$. Note that 2-dmKdV admits another solution given by

$$a_n = -\mathrm{i}B_2\sqrt{\frac{1 - m^2\mathrm{sn}(k,m)^2 - \mathrm{cn}(k,m)^2\left(1 - A_2^2 - B_2^2\right)}{\left(A_2^2 + B_2^2\right)(1 - m^2\mathrm{sn}(k,m)^2 - \mathrm{cn}(k,m)^2)}} + A_2\frac{\mathrm{sn}(kn + \omega t, m)}{\mathrm{cn}(kn + \omega t, m)}, \tag{23a}$$

$$b_n = \mathrm{i}A_2\sqrt{\frac{1 - m^2\mathrm{sn}(k,m)^2 - \mathrm{cn}(k,m)^2\left(1 - A_2^2 - B_2^2\right)}{\left(A_2^2 + B_2^2\right)(1 - m^2\mathrm{sn}(k,m)^2 - \mathrm{cn}(k,m)^2)}} + B_2\frac{\mathrm{sn}(kn + \omega t, m)}{\mathrm{cn}(kn + \omega t, m)}, \tag{23b}$$

where $\omega = -\frac{2(A_2^2+B_2^2)\mathrm{cn}(k,m)\mathrm{sn}(k,m)}{1-m^2\mathrm{sn}(k,m)^2-\mathrm{cn}(k,m)^2}$ and $m \neq 1$.

5. Summary

In this paper, we have shown that the 2-coupled discrete modified Korteweg–de Vries equation (2-dmKdV) governed by nonlinear partial differential–difference equations is completely integrable. We have derived Lax matrices for 2-dmKdV and also shown explicitly that it admits a sequence of generalized (non-point) symmetries and polynomial conserved densities. Also exact solutions of it expressible in terms of Hyperbolic and Jacobian elliptic functions have been derived.

The above analysis indicates that the N-coupled discrete equation (N-dmKdV),

$$\frac{\partial a_{i(n)}}{\partial t} = \left(1 + \sum_{j=1}^{N} a_{j(n)}^2\right)(a_{i(n+1)} - a_{i(n-1)}), \qquad i = 1, 2, \ldots, N, \tag{24}$$

can be viewed as a discrete analogue of the N-coupled modified Korteweg–de Vries equation

$$u_{i\tau} + 6\left(\sum_{j=1}^{N} u_j^2\right)u_{ix} + u_{ixxx}, \qquad i = 1, 2, \ldots, N. \tag{25}$$

It is straightforward to check that the coupled N-dmKdV given in equation (24) admits a sequence of commutable generalized symmetries. For example, the trivial generalized symmetry with rank $\frac{3}{2}$ is

$$G_2^{(i)}(n) = \left(1 + \sum_{j=1}^{N} a_{j(n)}^2\right)(a_{i(n+1)} - a_{i(n-1)}), \qquad i = 1, 2, \ldots, N,$$

and the next non-trivial generalized symmetry with rank $\frac{5}{2}$ can be computed whose explicit form is

$$G_3^{(i)}(n) = \left(1 + \sum_{j=1}^{N} a_{j(n)}^2\right)\left[\left(1 + \sum_{j=1}^{N} a_{j(n+1)}^2\right) a_{i(n+2)} - \left(1 + \sum_{j=1}^{N} a_{j(n-1)}^2\right) a_{i(n-2)} + \left(a_{i(n+1)}^2 - a_{i(n-1)}^2 - \sum_{j=1}^{N}\left(a_{j(n+1)}^2 - a_{j(n-1)}^2\right)\right) a_{i(n)} + 2\left(a_{j(n+1)}^2 - a_{j(n-1)}^2\right)\left(\sum_{j=1}^{N} a_{j(n)}\, a_{j(n+1)}\right)\right], \qquad i = 1, 2, \ldots, N.$$

In a similar manner, one can derive a sequence of next order generalized symmetries. Also one can show that the coupled N-dmKdV admits a sequence of polynomial conserved densities.

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References

[1] Toda M 1981 *Theory of Lattices* (Berlin: Springer)
[2] Ablowitz M J and Ladik F J 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213
[3] Ablowitz M J and Ladik F J 1976 Nonlinear differential–difference equations and Fourier analysis *J. Math. Phys.* **17** 1011
[4] McMillan E M 1971 *Topics in Physics* ed W E Brittin and H Odabasi (Boulder: Colorado Associated University Press)
[5] Quispel G R W, Roberts J A G and Thompson C J 1988 Integrable mappings and soliton equations *Phys. Lett.* A **126** 419
[6] Grammaticos B, Ramani A and Papageorgiou V G 1991 Do integrable mappings have the Painleve property? *Phys. Rev. Lett.* **67** 1825
[7] Capel H W, Nijhoff F W and Papageorgiou V G 1991 Complete integrability of Lagrangian mappings and lattices of KdV type *Phys. Lett.* A **155** 371
[8] Quispel G R W, Capel H W, Papageorgion V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica* A **173** 243
[9] Bruschi M, Ragnisco O, Santini P M and Gui-Zhang T 1991 Integrable symplectic maps *Physica* D **49** 273
[10] Veselov A P 1991 Integrable maps *Russ. Math. Surv.* **46** 1
[11] Joshi N, Burtonclay D and Halburd R G 1992 Nonlinear nonautonomous discrete isomonodromy problem *Lett. Math. Phys.* **26** 123
[12] Hietarinta J, Ramani A and Grammaticos B 1994 Continuous vacua in bilinear soliton equations *J. Phys. A: Math. Gen.* **27** 3149
[13] Grammaticos B and Ramani A 2004 Discrete Painlevé equations: a review. Discrete integrable systems *Lecture Notes Phys.* **644** 245
[14] Grammaticos B, Schwarzbach Y K and Tamizhmani K M 2004 *Discrete integrable systems (Lectures Notes in Physics* vol 644*)* (Berlin: Springer)
[15] Hay M, Hietarinta J, Joshi N and Nijhoff F 2007 A Lax pair for a lattice modified KdV equation, reductions to q-Painleve equations and associated Lax pairs *J. Phys. A: Math. Theor.* **40** F61
[16] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[17] Fokas A S 1987 Symmetries and integrability *Stud. Appl. Math.* **77** 253
[18] Shabat A B and Yamilov R I 1991 Symmetries of nonlinear chains *Lening. Math. J.* **2** 377
[19] Ablowitz M J and Clarkson P A 1991 *Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[20] Kac M and van Moerbeke P 1975 On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices *Adv. Math.* **16** 160

[21] Hirota R and Iwao M 2000 Time-discretization of soliton equations *Symmetries and Integrability of Difference Equations-III CRM Proc. Lecture Notes* eds D Levi and O Ragnisco (Providence, RI: American Mathematical Society) pp 217–29

[22] Suris Yu B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* vol 219 (Basle: Birkhäuser)

[23] Khare A, Dmitriev S V and Saxena A 2007 Exact static solutions of a generalized discrete ϕ^4model including short-periodic solutions arXiv:nlin/07101460

[24] Kevrekidis P G 2003 On a class of discretizations of Hamiltonian nonlinear partial differential equations *Physica* D **183** 68

[25] Ablowitz M J, Ohta Y and Trubatch A D 1999 On discretizations of the vector nonlinear Schrödinger equation *Phys. Lett.* A **253** 287

[26] Jia M, Liu P and Lou S Y 2008 An integrable coupled hybrid lattice and its exact solutions *J. Math. Phys.* **49** 072901–1

[27] Sahadevan R and Balakrishnan S 2008 Similarity reduction, generalized symmetries, recursion operator, and integrability of coupled Volterra system *J. Math. Phys.* **49** 113510

[28] Iwao M and Hirota R 1997 Soliton solutions of a coupled modified Korteweg–de Vries equations *J. Phys. Soc. Japan* **66** 577

[29] Tsuchida T and Wadati M 1998 The coupled modified Korteweg–de Vries equations *J. Phys. Soc. Japan* **67** 1175

[30] Sahadevan R and Kannagi N 2002 The coupled modified Korteweg–de Vries equations: similarity reduction, Lie–Backlund symmetries and integrability *J. Math. Phys.* **43** 3133

[31] Oevel W, Zhang H and Fuchssteiner B 1989 Master symmetries and multi-Hamiltonian formulations for some integrable lattice systems *Prog. Theor. Phys.* **81** 294

[32] Hereman W, Sanders J A, Sayers J and Wang J P 2005 Symbolic computation of polynomial conserved densities, generalized symmetries and recursion operators for nonlinear differential–difference equations *Group Theory and Numerical Analysis (CRM Proc. and Lecture Ser.* vol 39*)* ed P Winternitz *et al* (Providence, RI: American Mathematical Society) pp 267–82

[33] Olver P J 1977 Evolution equations possessing infinitely many symmetries *J. Math. Phys.* **18** 1212

[34] Baldwin D E and Hereman W 2009 A symbolic algorithm for computing recursion operators of nonlinear PDEs *Int. J. Comput. Math.* at press

[35] Göktas U, Hereman W and Erdmann G 1997 Computation of conserved densities for systems of nonlinear differential–difference equations *Phys. Lett.* A **236** 30

[36] Göktas U and Hereman W 1998 Computation of conserved densities for nonlinear lattices *Physica* D **123** 425

[37] Lakshmanan M and Tamizhmani K M 1985 Lie–Bäcklund symmetries of certain nonlinear evolution equations under perturbation around their solutions *J. Math. Phys.* **26** 1189